Int J Theor Phys (2010) 49: 1507–1515
DOI 10.1007/s10773-010-0331-4

Entanglement in the Extended XY Spin Model with Three Spin Interaction and External Field

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Received: 16 January 2010 / Accepted: 23 March 2010 / Published online: 1 April 2010

Abstract We investigate the pairwise thermal entanglement of the extended XY model with three spin interactions and external filed on zig-zag lattice. The influences of three spin interactions J_2 and external field λ on the thermal entanglement of the nearest neighbor (NN) and next nearest neighbor (NNN) spins are considered. It is found that J_2 and λ suppress both the maximal value and the critical temperature of the NN entanglement C_{12}. However, when it comes to the NNN entanglement C_{13}, there exists a critical value of J_2 above which both the maximal entanglement and the critical temperature can be enhanced by J_2 for a fixed external field. With J_2 fixed, the effect of λ on C_{13} are different for different values of J_2. For $J_2 < 1$, λ suppresses both T_C and the maximal values of C_{13}. For $J_2 \geq 1$, λ enhances the maximal values of C_{13} while decreases the critical temperature. These results show that one is able to get the entanglement wanted by properly controlling the values of the three spin interactions J_2 and the external field λ.

Keywords Quantum entanglement · Three spin interaction · Extended XY model

1 Introduction

Quantum entanglement is one of the most fascinating features of quantum mechanics [1, 2]. It has been considered as an useful resource for many quantum information processing protocols [3]. Such as quantum superdense coding [4], quantum teleportation [5], quantum

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computation [6] and quantum information [7]. The entanglement in solid state systems, which have been regarded as the most probable candidates to construct quantum computer, has received considerable attention [8–10]. In particular, the entanglement in spin chains are the subject of intense theoretical and experimental investigation in recent decades [11–16]. From the quantum information perspective, quantum spin chains have been proposed as quantum wires for short-distance quantum communication [17]. Moreover, the spin systems display quantum phase transitions (QPTs) [18], which is induced by the pure quantum fluctuations, at some critical points. It is expected that the entanglement of qubits may exhibit some special characters in the critical region of the spin system. In Ref. [19], the authors demonstrated that entanglement shows scaling behavior in the vicinity of the phase transition point.

On the other hand, there is a large number of references dedicated to the study of thermal entanglement in spin chains [20–26]. X.G. Wang studied the entanglement in the quantum Heisenberg XY model [27]. R. Liu et al. investigated the pairwise thermal entanglement in the XXX Heisenberg model. The influence of next nearest neighbor and nonuniform magnetic field on entanglement has been studied [28]. In most of these studies, attentions were focused on the two-body interactions, which are most readily accessible in experiments. However, the Hamiltonian with multi-spin interactions had been of limited interest as they were difficult to be implemented and controlled experimentally in the past. Recently, the experimental realization of three spin interactions in cold polar molecules [14] and atoms in optical lattices [15, 16] arouse researchers' interests. Tsomokos et al. studied the chiral entanglement in triangular lattice and found that the entanglement depends on the lattice geometry due to frustration effects [29]. M.F. Yang investigated the ground state entanglement properties of spin chains with three-body interactions [30]. To our knowledge, there is no report on the thermal entanglement in extended XY spin chains with three spin interactions [31].

In this article, we focus our attention on the study of thermal entanglement of the extended XY model with three spin interactions. The influences of three spin interactions and the external field on the thermal entanglement of nearest neighbor (NN) and next nearest neighbor (NNN) spins are studied. The rest of this paper is organized as follows. In Sect. 2 the model is introduced; In Sect. 3, we calculate and discuss the concurrence of thermal entanglement; Conclusion is given in Sect. 4.

2 Model

The Hamiltonian for the extended XY model with three spin interactions is given by

$$H = -J_1 \sum_l (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y) - J_2 \sum_l (\sigma_{l-1}^x \sigma_l^z \sigma_{l+1}^x + \sigma_{l-1}^y \sigma_l^z \sigma_{l+1}^y) + \lambda \sum_l \sigma_l^z \tag{1}$$

and describes the spin system determined on the zig-zag chain (see Fig. 1). In the Hamiltonian (1), J_1 is the interaction parameter between the nearest neighbor spins. $J_1 > 0$ corresponds to the anti-ferromagnetic interaction, while $J_1 < 0$ corresponds to the ferromagnetic interaction. In the calculation, we take $|J_1| = 1$ and all the other parameters are scaled by $|J_1|$, so that the parameters are dimensionless. J_2 is the three spin interactions, λ is the external field, σ_l^α $(\alpha = x, y, z)$ are the Pauli matrices, l indicates the location of spin.

In this paper, we take out a three-spin cluster and investigate the pairwise thermal entanglement on it. Since entanglement in a system with fewer spins may display general features of entanglement of system with more spins. Thus the study of pairwise entanglement in the

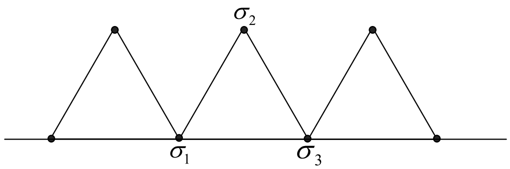

Fig. 1 Schematic representation of the structure of the extended XY model

three-spin cluster is meaningful [32]. The Hamiltonian of the three spin system is

$$H = -J_1 \sum_{l=1}^{3} (\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y) - J_2(\sigma_1^x \sigma_2^z \sigma_3^x + \sigma_1^y \sigma_2^z \sigma_3^y) + \lambda \sum_{l=1}^{3} \sigma_l^z \tag{2}$$

In the standard basis $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$, we obtain the eigenvalues of the above Hamiltonian (2) as

$$\begin{aligned}
E_{0,1} &= \mp 3\lambda, \\
E_{2,3} &= \mp(\lambda \pm 2J_2), \\
E_{4,5} &= \mp\big(-\lambda + J_2 \pm \sqrt{8J_1^2 + J_2^2}\big), \\
E_{6,7} &= \mp\big(-\lambda + J_2 \mp \sqrt{8J_1^2 + J_2^2}\big).
\end{aligned} \tag{3}$$

The corresponding eigenvectors are explicitly given by

$$\begin{aligned}
|\psi_0\rangle &= |000\rangle, \\
|\psi_1\rangle &= |111\rangle, \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}(|001\rangle - |100\rangle), \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}(|011\rangle - |110\rangle), \\
|\psi_4\rangle &= \frac{1}{\sqrt{2 + A^2}}(|110\rangle + A|101\rangle + |011\rangle), \\
|\psi_5\rangle &= \frac{1}{\sqrt{2 + B^2}}(|100\rangle + B|010\rangle + |001\rangle), \\
|\psi_6\rangle &= \frac{1}{\sqrt{2 + B^2}}(|110\rangle - B|101\rangle + |011\rangle), \\
|\psi_7\rangle &= \frac{1}{\sqrt{2 + A^2}}(|100\rangle - A|010\rangle + |001\rangle),
\end{aligned} \tag{4}$$

where $A = (-J_2 + \sqrt{8J_1^2 + J_2^2})/2J_1$ and $B = (J_2 + \sqrt{8J_1^2 + J_2^2})/2J_1$.

3 Thermal Entanglement

Our aim is to investigate the effects of the three spin interactions J_2 and external field λ on the pairwise entanglement. To qualify the entanglement we use the Wootters concurrence [33], defined as

$$C(\rho) = 2\max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}) \tag{5}$$

where λ_i $(i = 1, \dots, 4)$ are the eigenvalues in a decreasing order of the operator

$$R = \rho_{12}(\sigma_A^y \otimes \sigma_B^y)\rho_{12}^*(\sigma_A^y \otimes \sigma_B^y). \tag{6}$$

In the above (6), ρ_{12}^* denotes the complex conjugate of ρ_{12} and $\sigma_{A,B}^y$ are the Pauli matrices for qubits A and B. This quality attains its maximum value 1 for maximally entangled state and vanishes for separable state.

The state of the system at thermal equilibrium can be described as

$$\rho(T) = \frac{1}{Z}\sum_{n=0}^{7} \exp(-\beta E_n)|\psi_n\rangle\langle\psi_n| \tag{7}$$

where $Z = \mathrm{Tr}(\exp(-\beta H))$ is the partition function, $\beta = 1/k_B T$ with k_B the Boltzmann constant. In the calculation, we usually take $k_B = 1$ for simplicity. From (3), the analytical expression of Z can be obtained

$$Z = 4\,\mathrm{Cosh}\left(\frac{\lambda - J_2}{T}\right)\left(\mathrm{Cosh}\left(\frac{2\lambda + J_2}{T}\right) + \mathrm{Cosh}\left(\frac{\sqrt{8J_1^2 + J_2^2}}{T}\right)\right). \tag{8}$$

After tracing out the freedom of spin 3 in (7), we have the reduced density matrix for spins 1 and 2 in the standard basis of $|0_1 0_2\rangle$, $|0_1 1_2\rangle$, $|1_1 0_2\rangle$ and $|1_1 1_2\rangle$

$$\rho_{12}(T) = \frac{1}{Z}\begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{9}$$

The analytical expressions of the non-zero matrix elements of $\rho_{12}(T)$ are as follows

$$\begin{aligned}
\rho_{11} &= \exp(-E_0/T) + \frac{1}{2}\exp(-E_2/T) + \frac{1}{2+A^2}\exp(-E_5/T) + \frac{1}{2+B^2}\exp(-E_7/T), \\
\rho_{22} &= \frac{1}{2}\exp(-E_3/T) + \frac{1}{2+A^2}\exp(-E_4/T) + \frac{B^2}{2+B^2}\exp(-E_5/T) \\
&\quad + \frac{1}{2+B^2}\exp(-E_6/T) + \frac{A^2}{2+A^2}\exp(-E_7/T), \\
\rho_{33} &= \frac{1}{2}\exp(-E_2/T) + \frac{A^2}{2+A^2}\exp(-E_4/T) + \frac{1}{2+B^2}\exp(-E_5/T) \\
&\quad + \frac{B^2}{2+B^2}\exp(-E_6/T) + \frac{1}{2+A^2}\exp(-E_7/T), \\
\rho_{44} &= \exp(-E_1/T) + \frac{1}{2}\exp(-E_3/T) + \frac{1}{2+A^2}\exp(-E_4/T) + \frac{1}{2+B^2}\exp(-E_6/T), \\
\rho_{23} &= \rho_{32} = \frac{A^2}{2+A^2}\exp(-E_4/T) + \frac{B^2}{2+B^2}\exp(-E_5/T) \\
&\quad - \frac{B}{2+B^2}\exp(-E_6/T) - \frac{A}{2+A^2}\exp(-E_7/T).
\end{aligned} \tag{10}$$

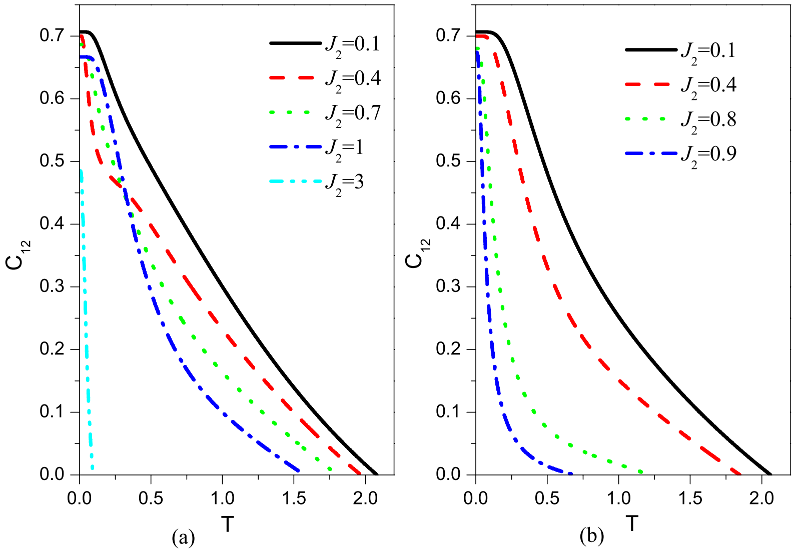

Fig. 2 The nearest neighbor concurrence C_{12} as a function of the temperature T with different three spin interactions J_2. The external fields are assumed to be $\lambda = 0.5$ (**a**) and $\lambda = 1$ (**b**)

From (5), (7) and (9), the concurrence for spins 1 and 2 is obtained as

$$C_{12} = \max\left\{2\max\frac{1}{Z}(\sqrt{\rho_{11}\rho_{44}}, |\rho_{23} - \sqrt{\rho_{22}\rho_{33}}|, |\rho_{23} + \sqrt{\rho_{22}\rho_{33}}|) - \frac{1}{Z}(2\sqrt{\rho_{11}\rho_{44}} + |\rho_{23} - \sqrt{\rho_{22}\rho_{33}}| + |\rho_{23} + \sqrt{\rho_{22}\rho_{33}}|)\right\}. \tag{11}$$

Substituting (10) into (11), one is able to analyze the entanglement behavior following the change of various parameters J_2 and λ.

In Fig. 2, we give the concurrence C_{12} versus temperature T curves for different three spin interactions J_2. The external fields are set to be $\lambda = 0.5$ (a) and $\lambda = 1$ (b), respectively. It can be known from the figures that the concurrence decreases monotonically with T and finally reaches to zero at a critical temperature T_C. The critical temperature T_C tends to be suppressed with the increasing of J_2. In addition, one can notice from the figure that for a given J_2, there exists a maximal value for C_{12}. But the influences of J_2 on these maximal values are difference for $\lambda = 0.5$ and $\lambda = 1$. For $\lambda = 0.5$, J_2 greatly degrades the maximal value (see (a) for detail). When it comes to $\lambda = 1$, J_2 affects little to the maximal entanglement (see (b) for detail). Our numerical calculation also shows that for $\lambda = 1$ and $J_2 \geq 1$, the concurrence of the nearest neighbor system completely disappears, i.e., $C_{12} = 0$. The change of maximum C_{12} with J_2 is due to the mixture of the ground state. From (3), we can know that for a given λ, J_2 changes the eigenvalues of the system. Consequently, it changes the mixture of the ground state and the maximal value of C_{12}.

The concurrence C_{12} as the function of T for different external fields λ is plotted in Fig. 3. It is noticed that the influences of external field on C_{12} are difference for $J_2 = 0.6$ and $J_2 = 1.8$. For $J_2 = 0.6$ (Fig. 3(a)), the concurrence firstly monotonically decreases with T when the external field is small (e.g. $\lambda = 0.1, 0.5, 0.9$). But when $\lambda \geq 1.2$, the concurrence exhibits a smooth revival as the temperature increases until a maximum value, then collapses gradually to zero (e.g. $\lambda = 1.2$ and 2). The maximal value is suppressed with the increasing

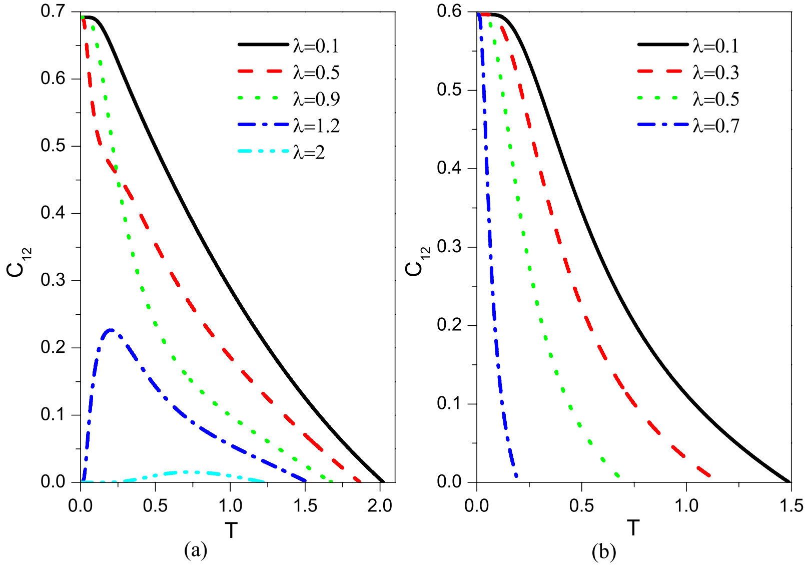

Fig. 3 The nearest neighbor concurrence C_{12} as a function of the temperature T with different external fields λ. The three spin interactions are assumed to be $J_2 = 0.6$ (**a**) and $J_2 = 1.8$ (**b**)

of λ. It is also noteworthy that the critical temperature T_C decreases as the external field λ increases. The C_{12} versus T curves for $J_2 = 1.8$ with different fixed external fields are plotted in Fig. 3(b). As can be seen from the figure that C_{12} decays monotonically with T. There is no revival phenomenon which differs from that in Fig. 3(a). Meanwhile, the critical temperature T_C decreases as the increasing of the external field λ.

Now, we turn to study the entanglement of the next nearest neighbor spins 1 and 3. After tracing out the freedom of spin 2 in (7), we obtain the reduced density matrix for spins 1 and 3. In the basis $|0_1 0_3\rangle$, $|0_1 1_3\rangle$, $|1_1 0_3\rangle$ and $|1_1 1_3\rangle$, we have the matrix form of ρ_{13}

$$\rho_{13}(T) = \frac{1}{Z}\begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
\rho_{11} &= \exp(-E_0/T) + \frac{B^2}{2+B^2}\exp(-E_5/T) + \frac{A^2}{2+A^2}\exp(-E_7/T), \\
\rho_{22} = \rho_{33} &= \frac{1}{2}[\exp(-E_2/T) + \exp(-E_3/T)] + \frac{1}{2+A^2}[\exp(-E_4/T) + \exp(-E_7/T)] \\
&\quad + \frac{1}{2+B^2}[\exp(-E_5/T) + \exp(-E_6/T)], \qquad (13) \\
\rho_{23} = \rho_{32} &= -\frac{1}{2}[\exp(-E_2/T) + \exp(-E_3/T)] + \frac{1}{2+A^2}[\exp(-E_4/T) + \exp(-E_7/T)] \\
&\quad + \frac{1}{2+B^2}[\exp(-E_5/T) + \exp(-E_6/T)].
\end{aligned}$$

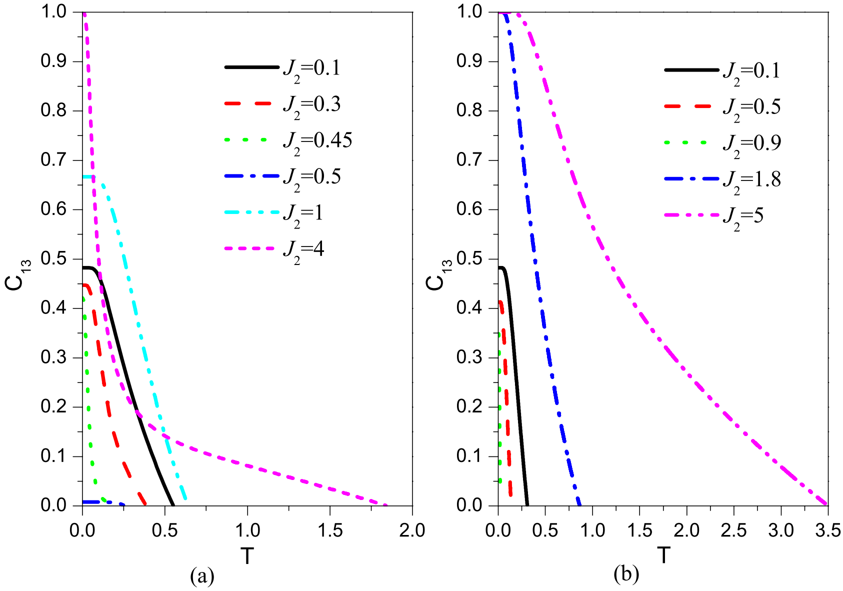

Fig. 4 The next nearest neighbor concurrence C_{13} as a function of the temperature T with different three spin interactions J_2. The external fields are assumed to be $\lambda = 0.5$ (**a**) and $\lambda = 1$ (**b**)

Similarly, from (5), (7) and (12), we can obtain the concurrence of spins 1 and 3

$$C_{13} = \max\left\{2\max\frac{1}{Z}(\sqrt{\rho_{11}\rho_{44}}, |\rho_{22}-\rho_{33}|, |\rho_{22}+\rho_{33}|) - \frac{1}{Z}(2\sqrt{\rho_{11}\rho_{44}} + |\rho_{22}-\rho_{33}| + |\rho_{22}+\rho_{33}|)\right\}. \quad (14)$$

Substituting (13) into (14), one can obtain the analytical expression of C_{13}. The influence of the three spin interactions J_2 and external field λ on C_{13} are as follows.

In Fig. 4, we plot C_{13} as the function of the system temperature T with different three spin interactions J_2. It is showed that the concurrence decays monotonically with T and reaches to zero at a finite temperature T_C just as that in Fig. 1. The three spin interactions J_2 firstly suppresses T_C and the maximal values of C_{13} for both $\lambda = 0.5$ and $\lambda = 1$. However, as J_2 becomes larger than an optimal value both T_C and the maximal value of C_{13} increase sharply. Particularly, for $J_2 \geq 4$ (a) and $J_2 \geq 1.8$ (b), the maximal value of C_{13} equals to 1. The critical temperature T_C increases with further increases of J_2.

In Fig. 5, we plot C_{13} as the function of temperature T for different external fields λ. We can know from the figure that given an external field the concurrence decays monotonically with T for $J_2 = 0.6$ and $J_2 = 1.8$. But the influence of λ on C_{13} are different. As to $J_2 = 0.6$ (a), the maximal value of C_{13} and the critical temperature T_C are suppressed by the increasing of λ which is consistent with that in Fig. 2. As to $J_2 = 1.8$ (b), T_C firstly decreases for $\lambda \in (0, 1)$. As λ becomes larger than a critical value both C_{13} and T_C appear sudden jump phenomenon (e.g. $\lambda = 1.2$ and $\lambda = 1.6$). With further increases of λ, T_C tends to decrease.

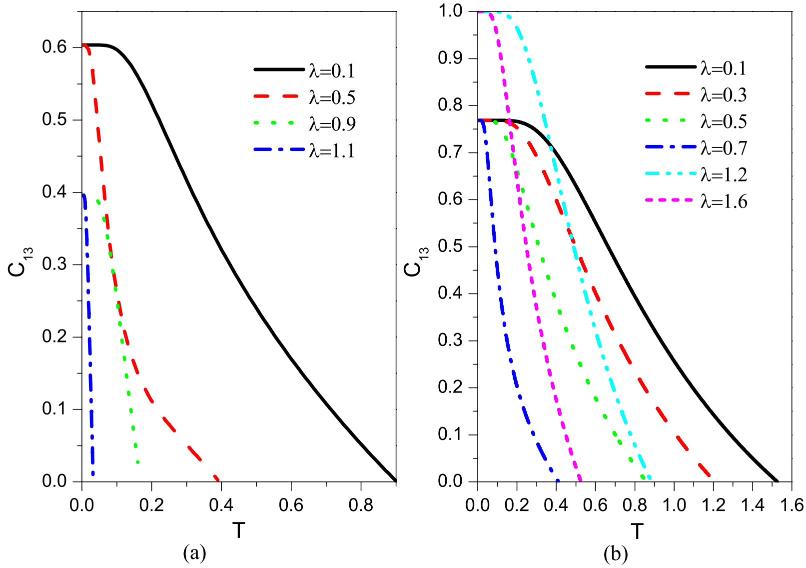

Fig. 5 The next nearest neighbor concurrence C_{13} as a function of the temperature T with different external fields λ. The three spin interactions are assumed to be $J_2 = 0.6$ (**a**) and $J_2 = 1.8$ (**b**)

4 Conclusion

To conclude, we have investigated thermal entanglement of the extended XY model with three spin interactions and external filed on zig-zag lattice. The influences of three spin interactions J_2 and external field λ on the thermal entanglement of the nearest neighbor (NN) and next nearest neighbor (NNN) spins are investigated. Our results show that J_2 and λ suppress both the maximal value and the critical temperature of the NN entanglement (C_{12}). However, when it comes to the NNN entanglement C_{13}, there exists a critical value of J_2 above which both the maximal entanglement and the critical temperature can be enhanced by J_2 for a fixed external field. With J_2 fixed, the effects of λ on C_{13} are different for different values of J_2. For $J_2 < 1$, λ suppresses both T_C and the maximal values of C_{13}. For $J_2 \geq 1$, λ enhances the maximal values of C_{13} while decreases the critical temperature. These results show that one is able to get the entanglement wanted by properly controlling the values of the three spin interactions J_2 and the external field λ.

Acknowledgements This work is supported by the National Natural Science Foundation of China under Grant No. 10904033, Natural Science Foundation of Hubei Province Grant No. 2009CDA145, Educational Commission of Hubei Province under Grant No. D20092204 and the Postgraduate Programme of Hubei Normal University under Grant No. 2007D20.

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